The `isl` Scheduler

Sven Verdoolaege

KU Leuven and Polly Labs

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Outline

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Overview of isl

isl is a thread-safe C library for manipulating **integer sets and relations**

- bounded by *affine constraints*
- involving *symbolic constants* and
- *existentially quantified variables*

plus **quasi-affine** and **quasi-polynomial functions** on such domains

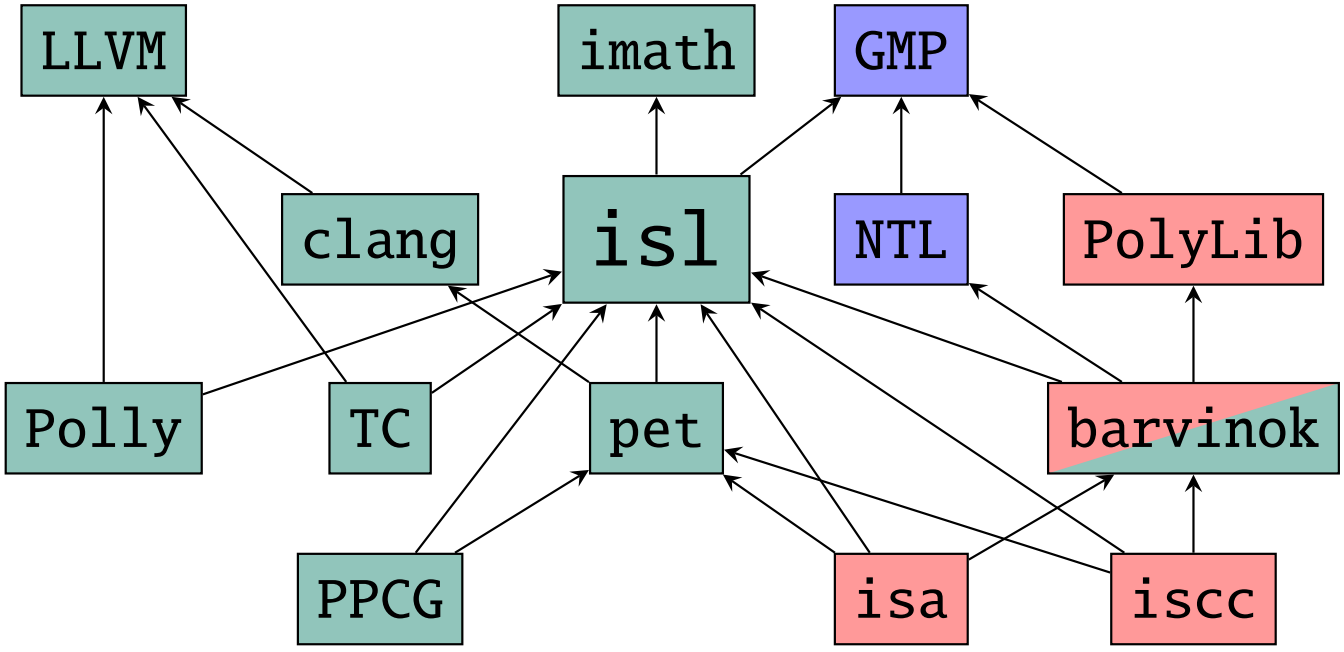
Supported operations by core library include

- *intersection*
- *union*
- *set difference*
- *integer projection*
- *coalescing*
- *closed convex hull*
- *sampling*, *scanning*
- *integer affine hull*
- *lexicographic optimization*
- *transitive closure* (approx.)
- *parametric vertex enumeration*
- *bounds on quasipolynomials*

Polyhedral compilation library

- *schedule trees*
- *dataflow analysis*
- *scheduling*
- *AST generation*

Connection with other Libraries and Tools



Licenses:
BSD/MIT/
Apache
LGPL
GPL

`isl`: manipulates parametric affine sets and relations
`barvinok`: counts elements in Presburger sets and relations
`pet`: extracts polyhedral model from clang AST
`PPCG`: Polyhedral Parallel Code Generator
`iscc`: interactive calculator
`TC`: Tensor Comprehensions

Set Representation

[12]

```
S:   A[0] = 1;
     for (i = 1; i < N; ++i)
T:       A[i] = 2 * A[i - 1];
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[N] -> { S[]; T[i]: 1 <= i < N }
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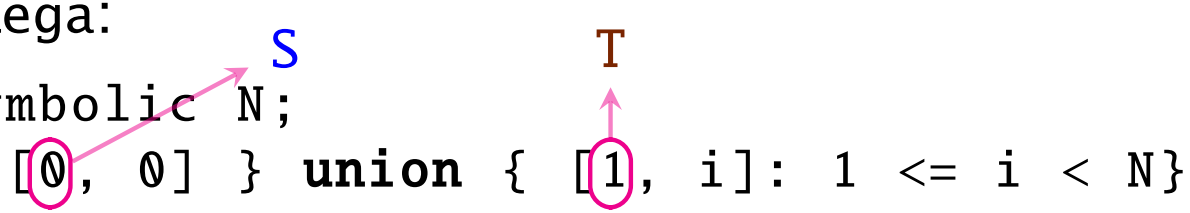

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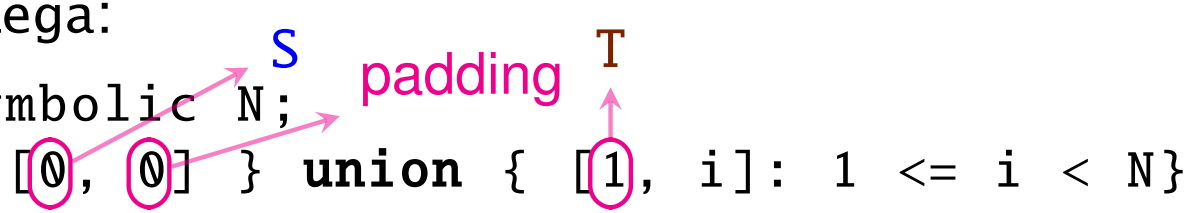

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S, padding, T

- PolyLib: (deals with rational sets, polyhedra)

```
2
2 5
0   1   0   0   0
0   0   1   0   0
3 5
0   1   0   0  -1
1   0   1   0  -1
1   0  -1   1  -1
```

equality/inequality, N

Schedule Representation

[7, 9, 12, 15]

$$
\begin{aligned}
T_1 &: \{[i] \to [0, i]\} \\
T_2 &: \{[i, j] \to [1, j, 0, i]\} \\
T_3 &: \{[i] \to [1, i-1, 1]\}
\end{aligned}
$$

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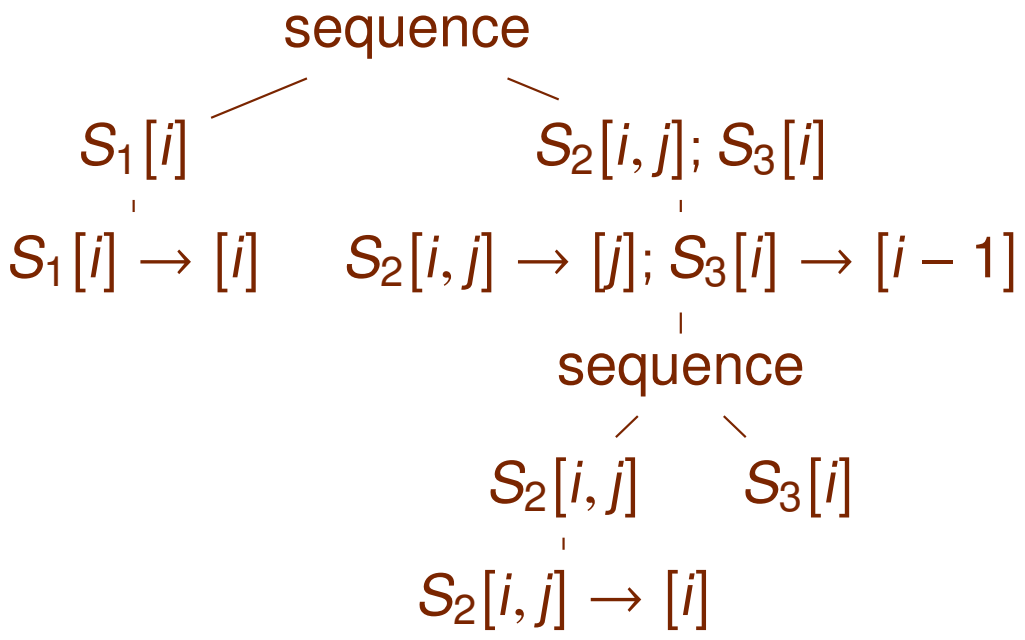
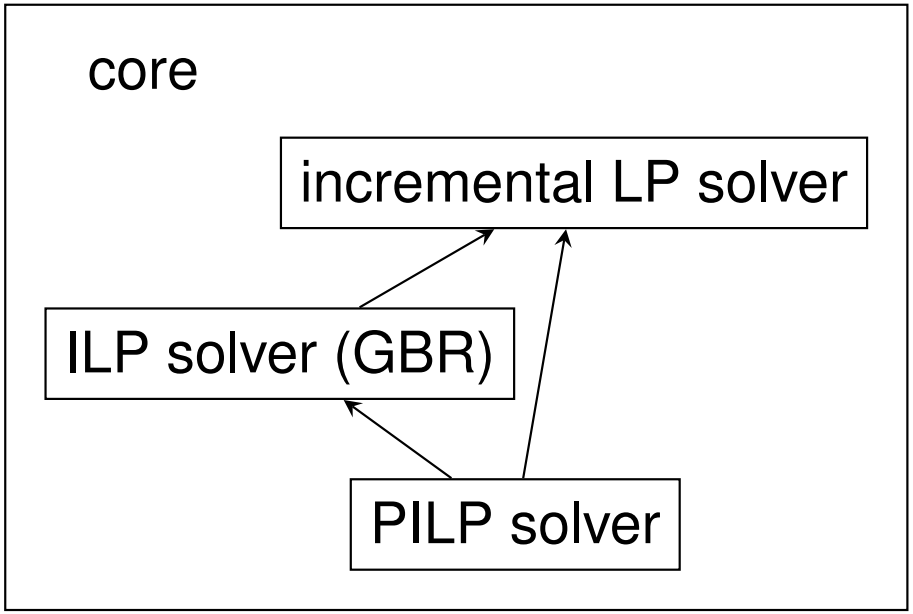


- Kelly's abstraction
  - schedule spread over statements
- union maps
  - single object
  - schedule transformations can be composed
  - flat schedule space (padding)
- schedule trees
  - single object
  - structured schedule space

[2, 3, 4, 10, 11, 16]

Internal Structure of isl



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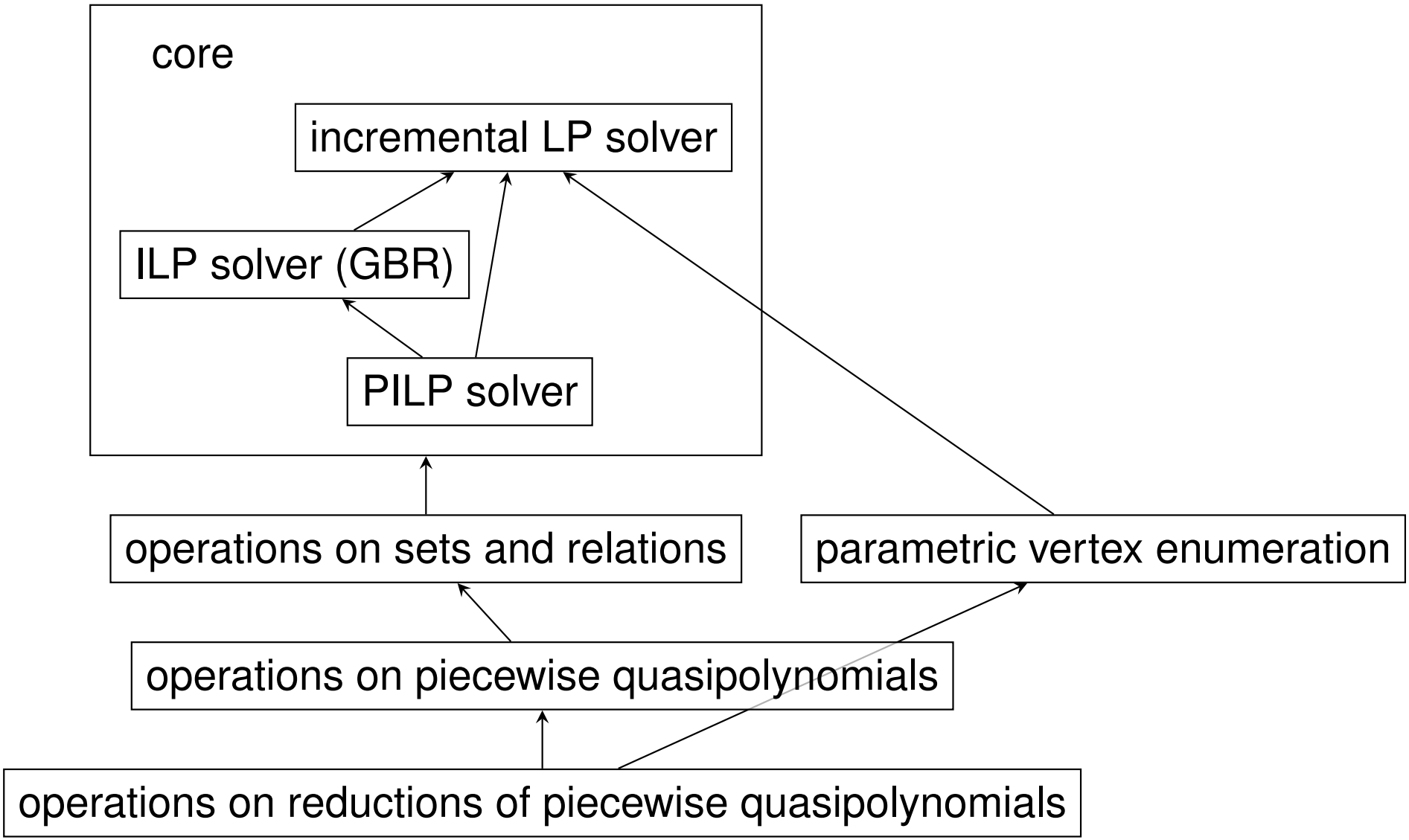

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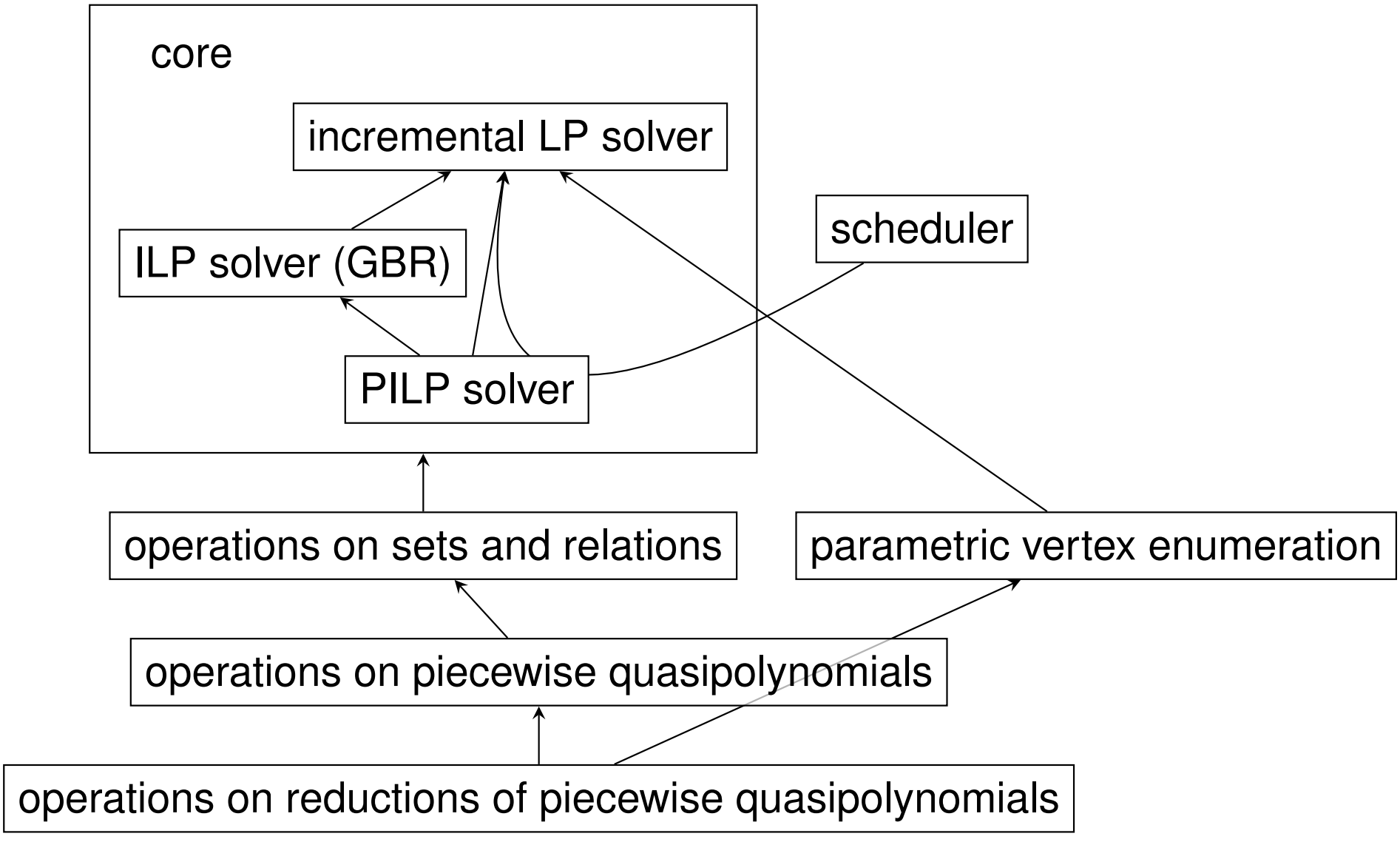

Internal Representation of Sets and Relations

Each set or relation is stored in disjunctive normal form (DNF)

$$R = \bigcup_i R_i$$

$$R_i = \{ S(\mathbf{i}) \to T(\mathbf{j}) : \exists \mathbf{k} : A_0\mathbf{c} + A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k} \geq \mathbf{a} \}$$

Each disjunct consists of

- affine equalities and inequalities
- symbolic constants **c**
- local variables **k**
  - existentially quantified, or,
  - integer division $k_i = \lfloor e_i / d_i \rfloor$

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Conversion to DNF

$$\neg(\exists \mathbf{a} : f(\mathbf{x}, \mathbf{a})) \quad \to \quad \neg f(\mathbf{x}, g(\mathbf{x}))$$

$\Rightarrow$ determine unique value of **a** satisfying $f(\mathbf{x}, \mathbf{a})$ and write it as an explicit piecewise quasi affine expression $g(\mathbf{x})$ of **x**

$\Rightarrow$ using parametric integer linear programming

Outline

Scheduling Constraints

[13]

- Validity $\mathbf{a} \rightarrow \mathbf{b}$
  - $\Rightarrow$ statement instance **b** needs to be executed after **a**
  - $\Rightarrow$ $f(\mathbf{b}) \geq f(\mathbf{a})$
- Proximity $\mathbf{a} \rightarrow \mathbf{b}$
  - $\Rightarrow$ statement instance **b** preferably executed close to **a**
  - $\Rightarrow$ $f(\mathbf{b}) - f(\mathbf{a})$ as small as possible
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- Conditional validity (live-range reordering)
  - condition $\mathbf{b} \to \mathbf{c}$ ($\leftsquigarrow$ flow dependences)
  - conditioned validity $\mathbf{a} \to \mathbf{b}$, $\mathbf{c} \to \mathbf{d}$ ($\leftsquigarrow$ order dependences)

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Dependences and Schedule Constraints

[13]

Traditional dependences

- flow dependences
  - ⇒ validity constraints
  - ⇒ proximity constraints
  - ⇒ coincidence constraints (when parallelism is important)
- false dependences
  - ⇒ validity constraints
  - ⇒ coincidence constraints (when parallelism is important)
  - ⇒ proximity constraints (optional for memory reuse)
- pairs of reads with shared write (“input dependences”)
  - ⇒ proximity constraints (optional)

Live-range reordering

- somewhat different classification of dependences
- slightly different mapping to schedule constraints

For example, PPCG currently

- adds false dependences to proximity constraints for historical reasons
- does not consider input dependences
- uses live-range reordering by default

Schedule Output

[15]

Scheduler produces a schedule tree

Main node types

- *band*: instances are executed according to associated multi-dimensional piecewise quasi-affine partial schedule

  the elements of a band are called its *members*

- *sequence*: children are executed in order

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- *sequence*: children are executed in order
- *set*: children may be executed in any order

Scheduling Algorithms

[1, 5]

Scheduling algorithms available in `isl`:

- Feautrier
  - carry as many (groups of) validity constraints as possible
  - fine-grained parallelism
- Pluto-algorithm
  - tilability (through permutability)
  - locality with parallelism as extreme case

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Constraints:

- Validity $S_i[\mathbf{x}] \to S_j[\mathbf{y}]$: $f_j(\mathbf{y}) - f_i(\mathbf{x}) \geq 0$
  Farkas $\to$ constraints on $\mathbf{c}_i^{\mathrm{x}}$, $\mathbf{c}_i^{\mathrm{n}}$ and c_i^{c}

Feautrier Scheduler

[5, 16]

- Carry as many groups of validity constraints $S_i[\mathbf{x}] \rightarrow S_j[\mathbf{y}]$ as possible

$$f_j(\mathbf{y}) - f_i(\mathbf{x}) \geq e_k \qquad \text{with } 0 \leq e_k \leq 1$$

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- Small coefficients $\Rightarrow$ minimize $\sum_{ij} |c^{\mathrm{x}}_{i,j}|$ and $\sum_{ij} c^{\mathrm{n}}_{i,j}$

  Note $|c^{\mathrm{x}}_{i,j}|$ requires extra variables
  - $|c^{\mathrm{x}}_{i,j}| = b_{i,j}$ with $c^{\mathrm{x}}_{i,j} \leq b_{i,j} \wedge -c^{\mathrm{x}}_{i,j} \leq b_{i,j}$, or
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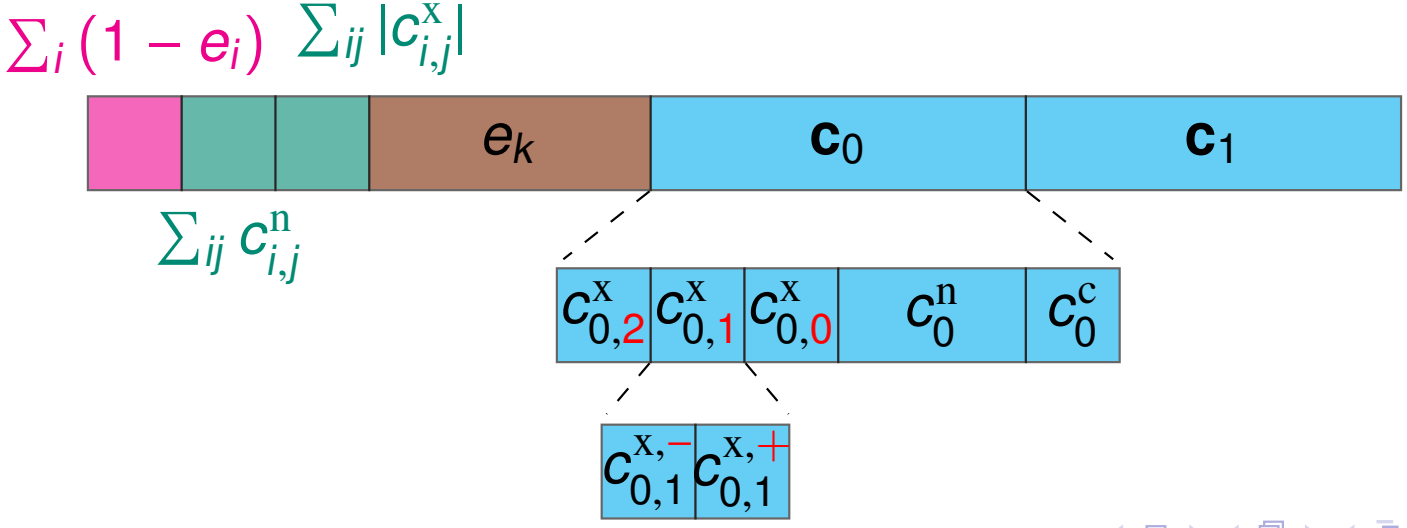

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- Proximity (temporal locality) $S_i[\mathbf{x}] \to S_j[\mathbf{y}]$: $f_j(\mathbf{y}) - f_i(\mathbf{x})$ small
  Farkas $\to$ constraints on $\mathbf{c}_i^{\mathrm{x}}$, $\mathbf{c}_i^{\mathrm{n}}$ and c_i^{c}
- Coincidence (parallelism) $S_i[\mathbf{x}] \to S_j[\mathbf{y}]$: $f_j(\mathbf{y}) - f_i(\mathbf{x}) = 0$
  Farkas $\to$ constraints on $\mathbf{c}_i^{\mathrm{x}}$, $\mathbf{c}_i^{\mathrm{n}}$ and c_i^{c}
- Linear independence of previous rows ($T_{i,0}$): $\mathbf{c}_i^{\mathrm{x}} \neq Y T_{i,0}$

  $\Rightarrow$ compute orthogonal complement of $T_{i,0}$: $U_i T_{i,0}^{\mathrm{t}} = \mathbf{0}$
  $\Rightarrow$ impose $U_i \mathbf{c}_i^{\mathrm{x}} \neq \mathbf{0}$

Pluto Scheduler

[1, 16]

- Uniform bound on proximity constraints $S_i[\mathbf{x}] \to S_j[\mathbf{y}]$:

$$u(\mathbf{n}) = \mathbf{m} \cdot \mathbf{n} + m_0$$

Impose $f_j(\mathbf{y}) - f_i(\mathbf{x}) \leq \mathbf{m} \cdot \mathbf{n} + m_0$ and minimize $\sum_k |m_k|$ and m_0

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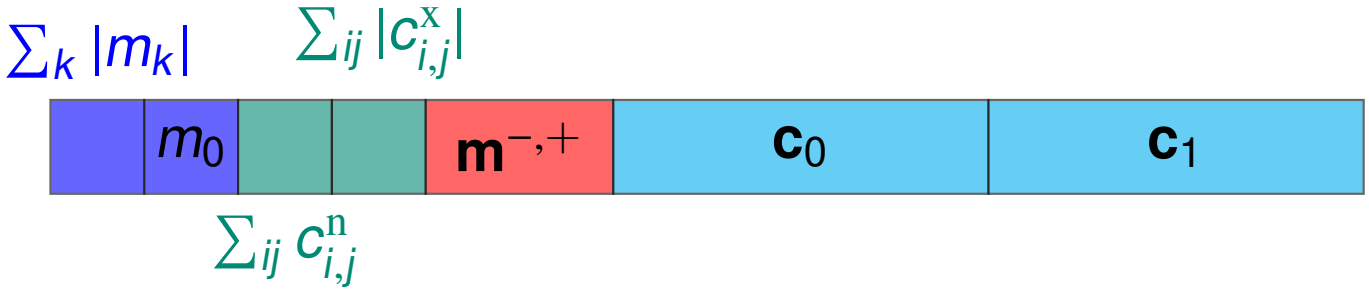

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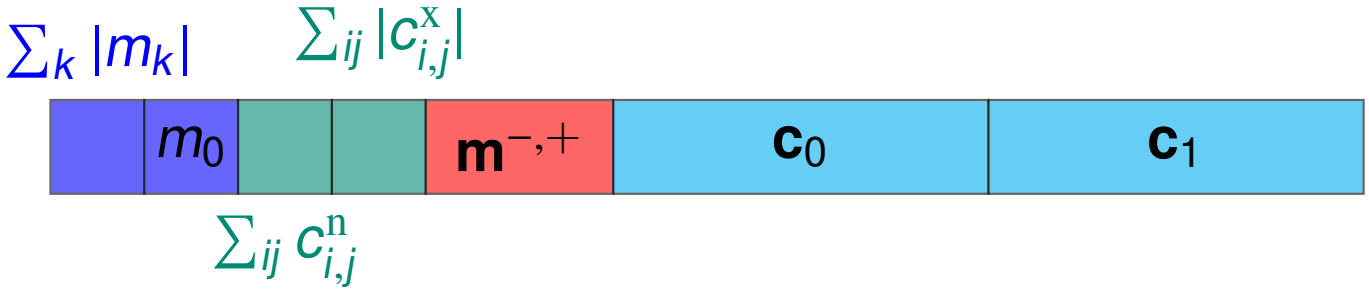

- Linear independence $U\,\mathbf{c}^{\mathrm{x}} \neq \mathbf{0}$
  - $\Rightarrow$ not linear
  - $\Rightarrow$ backtracking search: $U_i\mathbf{c}^{\mathrm{x}} \geq 1$ or $U_i\mathbf{c}^{\mathrm{x}} \leq -1$

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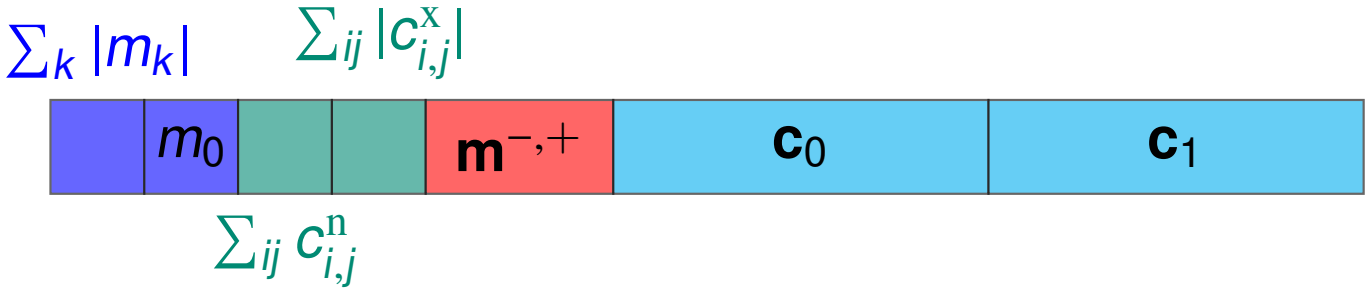

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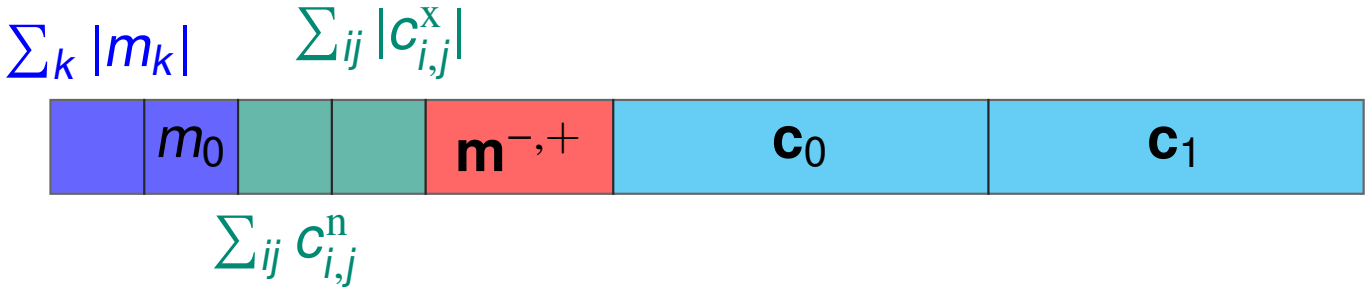

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- Coincidence constraints first imposed then relaxed
- Incremental scheduling
  1. First schedule SCCs separately
  2. Then combine SCCs incrementally

  $\Rightarrow$ better control over coincidence and band depth

Known Issues with Scheduling Algorithms

- Scheduling may take a long time
- Schedule may contain large coefficients
  - Schedule may result in loop coalescing
  - Schedule may be have large numerators (Feautrier)
  - Schedule may be unnecessarily scaled (Feautrier)
- Proximity constraints may affect feasibility (Pluto)

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Long Scheduling Times

- Possibly inherent to solution of large (I)LP problems
- Possibly related to Gomory's cutting plane method used in lexicographic dual simplex ILP solver [4]
  On some instances, ILP solver appears to get stuck in seemingly endless cycle

  Alternatives:
  - optimize each optimization criterion c_i in turn, or
  - pick large value M and solve for $\sum_i M^{n-i-1} c_i$

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Coalescing

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  for (int j = 0; j < 6; ++j)
S:  s += f(i, j);
```

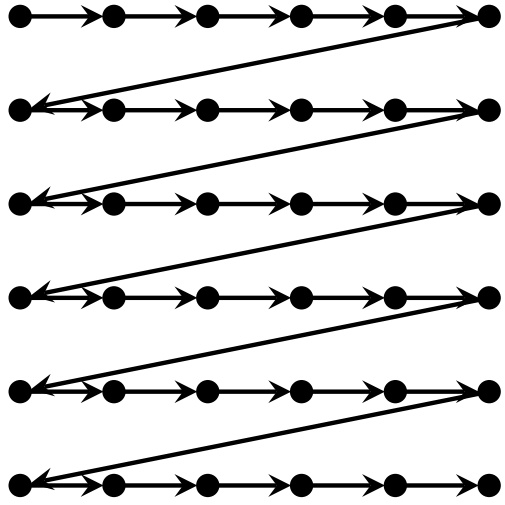

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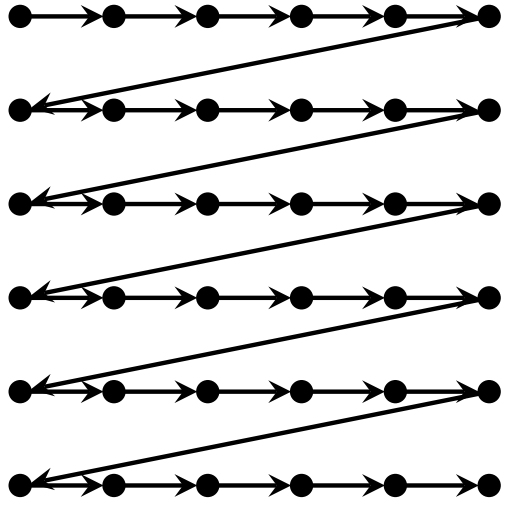

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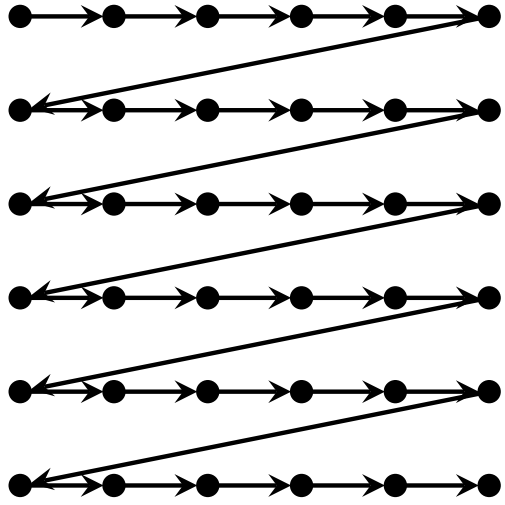

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Handling in `isl`:

- Pluto-algorithm (ILP)
  - $\Rightarrow$ impose bounds on coefficients based on instances set sizes
- Feautrier (LP)
  - $\Rightarrow$ detect coalescing in result and retry with smallest coefficient set to zero

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Scaled Schedules

Feautrier tends to schedule the n statements in an SCC apart

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In a first attempt, `isl` now only carries self-dependences
$\Rightarrow$ e_k only introduced for validity constraints from node to itself

If result is trivial
$\Rightarrow$ second attempt with all validity constraints

Known Issues with Scheduling Algorithms

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  - Schedule may be unnecessarily scaled (Feautrier)
- Proximity constraints may affect feasibility (Pluto)

Proximity Constraints

Recall:

- Proximity $\mathbf{a} \rightarrow \mathbf{b}$
  - $\Rightarrow$ statement instance **b** preferably executed close to **a**
  - $\Rightarrow$ $f(\mathbf{b}) - f(\mathbf{a})$ as small as possible

Proximity Constraints

Recall:

- Proximity $\mathbf{a} \to \mathbf{b}$
  - $\Rightarrow$ statement instance $\mathbf{b}$ preferably executed close to $\mathbf{a}$
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Pluto-algorithm

- looks for uniform bound $f(\mathbf{b}) - f(\mathbf{a}) \leq u(\mathbf{n})$ over all such pairs
- "minimizes" $u(\mathbf{n})$

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```
A:   a = f1();
     for (int i = 0; i < n; ++i)
B:       A[i] = a;
C:   b = f1();
     for (int i = 0; i < m; ++i)
D:       B[i] = b;
```

Proximity constraints: $\{ \mathtt{A}[] \to \mathtt{B}[i] : 0 \leq i < n; \mathtt{C}[] \to \mathtt{D}[i] : 0 \leq i < m \}$

- $\Rightarrow$ $u(\mathbf{n})$ needs to be larger than n and m
- $\Rightarrow$ $u(\mathbf{n})$ cannot involve m (constraint $\mathtt{A}[] \to \mathtt{B}[i]$ for every value of m)
- $\Rightarrow$ $u(\mathbf{n})$ cannot involve n (constraint $\mathtt{C}[] \to \mathtt{D}[i]$ for every value of n)
- $\Rightarrow$ no non-trivial solution

Proximity Constraints

Recall:

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Pluto-algorithm

- looks for uniform bound $f(\mathbf{b}) - f(\mathbf{a}) \leq u(\mathbf{n})$ over all such pairs
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Note: if some proximity constraint enforces large $u(\mathbf{n})$
then other proximity constraints are essentially ignored

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